The bifurcation of parabolic fixed points of complex dynamics and its applications
(複素力学系の放物型不動点の分岐とその応用)

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Introduction

Complex dynamics involves various types of bifurcations, which are mutually intertwined. Parabolic fixed points is one of the sources of drastic bifurcation, including the discontinuity of Julia set and the creation of subsets of large Hausdorff dimension. We also discuss the idea of renormalization in Dynamical systems, and applications of this idea for near-parabolic fixed points.

1 Parabolic fixed points and Fatou coordinates

1.1 Julia sets and Fatou sets

Definition (Fatou sets and Julia sets)**.** For a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$, define the Fatou set f and the Julia set j_f by:

$$F_f := \left\{ z \in \widehat{\mathbb{C}} \,\middle|\, \{f^n\}_{n=0}^{\infty} \text{ is normal in a neighborhood of } z \right\};$$
$$J_f := \widehat{\mathbb{C}} \smallsetminus F_f.$$

Connected components are called *Fatou components.* For a polynomial f of degree $f \geq 2$, define the filled-in Julia set K_p by

$$K_f := \{z \in \mathbb{C} \,|\, \{p^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

We have the following facts.

Proposition 1.1 (Fatou sets and Julia sets)**.** *The sets F_f, J_f, K_f are completely invariant in sense that $f(F_f) = F_f = f^{-1}(F_f)$, etc. J_f and K_f are closed and non-empty, F_f is open. For a polynomial f, $J_f = \partial K_f$.*

Theorem 1.2 (Sullivan's no wandering domain)**.** *Every Fatou component is eventually periodic.*

Theorem 1.3 (Classification of periodic Fatou components)**.** *If U is a periodic Fatou component of a rational map f with period p, then it is one of the following types: (i) attracting basin: orbits from U converges to a cycle of attracting periodic points in U; (ii) parabolic basin: orbits from U converges to a cycle of parabolic periodic points in ∂U; (iii) Siegel disk: $f^p|_U$ is conformally* *conjugate to an irrational rotation* $z \mapsto e^{2\pi i\alpha}z$ *on* $\mathbb{D} = \{z : |z| < 1\}$*; (iv) Herman:* $f^p|_U$ *is conformally conjugate to an irrational rotation* $z \mapsto e^{2\pi i\alpha}z$ *on* $A_r := \{z : r < |z| < 1\}$ $(0 < r < 1)$*;*

In this talk, we will focus on parabolic fixed points (especially the case $\lambda = 1$) and their bifurcation.

1.2 Fixed points

Definition (Periodic points, multiplier, classification)**.** Let f be a holomorphic function defined near $z_0 \in \mathbb{C}$. If z_0 is periodic point of period p (i.e. $f^p(z_0) = z_0$ and $f^j(z_0) \neq z_0$ $(1 \leq j < p)$, $\lambda = (f^p)'(z_0)$ is called *multiplier* of z_0. z_0 is called *attracting, indifferent, repelling* if $|\lambda| < 1, = 1, > 1$, respectively. When it is indifferent, write $\lambda = e^{2\pi i\alpha}$, and z_0 is called *parabolic, irrationally indifferent* if $\alpha \in \mathbb{Q}, \in \mathbb{R} \setminus \mathbb{Q}$ respectively.

If $\lambda(z_0) \neq 1$, the periodic point z_0 has a "continuation" as a periodic point. If $\lambda(z_0) = 1$, and $ord(f(z) - z, z_0) = m$, then it may bifurcate into m periodic points.

1.3 Normal forms and linearization

We want to simplify the map by conjugacies (coordinate changes).

Theorem 1.4 (Formal normal form)**.** *If* $f(z) = \lambda z + O(z^2)$ *and* λ *is not a root of unity, then it can be conjugated by a formal power series* $\varphi(z) = z + O(z^2)$ *to* $z \mapsto \lambda z$*.*

Definition (Linearization)**.** A fixed point z_0 of f is called *linearizable* if there exists a conformal map $\varphi(z)$ defined near z_0 such that $\varphi(z_0) = 0$, $\varphi'(z_0) \neq 0$ such that locally $\varphi \circ f = \lambda\varphi(z)$ near z_0.

Theorem 1.5 (Linearization)**.** *If* z_0 *is a fixed point of* f *such that* $|\lambda| < 1$ *or* $|\lambda| > 1$ *then it is linearizable.*

If f is a global map (rational function, entire function) and $|\lambda| < 1$, then the linearization map φ is extended to the attracting basin. If $|\lambda| > 1$, its inverse $\psi = \varphi^{-1}$ is extended to $\mathbb{C}$.

1.4 Fatou coordinates

Suppose f has a fixed point at $z = 0$ with multiplier $\lambda = 1$. Hence $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. We mainly discuss the non-degenerate case $a_2 \neq 0$. By a linear change of coordinate, we may assume $a_2 = 1$.

We change the coordinate change $w = -\frac{1}{z}$, we obtain a new map $w \mapsto F(w)$, for which $z = \infty$ is the parabolic fixed point.

$$\begin{aligned} w_1 = F(w) &= -\frac{1}{z_1} = -\frac{1}{z + z^2 + a_3z^3 + \cdots} = -\frac{1}{-\frac{1}{w} + \frac{1}{w^2} - a_3\frac{1}{w^3} + \cdots} \\ &= \frac{w}{1 - \frac{1}{w} + \frac{a_3}{w^2} + \cdots} = w\left(1 + \frac{1}{w} + \frac{1 - a_3}{w^2} + \cdots\right) \\ &= w + 1 + \frac{1 - a_3}{w} + \cdots \end{aligned}$$

Definition (Parabolic basin)**.** For a parabolic fixed point z_0 of f, its parabolic basin is defined as $\mathcal{B} = \mathcal{B}(z_0) = \{z : \ \{f^n\}_{n=0}^{\infty}$converges to z_0 locally uniformy near $z_0\}$.

Definition (Petals)**.** If $F(w) = w + 1 + o(1)$ near $w = \infty$, then $\{w : \operatorname{Re}(w) > R\}$ is forward invariant under F for large $R > 0$, this set is called *attracting petal* of F for $w = \infty$, and $\{w : \operatorname{Re}(w) < -R\}$ is backward invariant (*repelling petal*). These petals are transferred to the neighborhood of parabolic fixed point $z_0 \neq \infty$, by an appropriate Möbius coordinate change.

Definition (Fatou coordinates)**.** In attracting or repelling petals or other domains of f, a function Φ is called a *Fatou coordinate* if it satisfies $\Phi(f(z)) = \Phi(z) + 1 = T(\Phi(z))$. It it is defined if the attracting petal (resp. repelling petal) it is called *attracting Fatou coordinate* (reps. *repelling Fatou coordinate*). (or incoming/outgoing).

Theorem 1.6 (Existence of attracting/repelling Fatou coordinates)**.** *For a non-degenerate 1-parabolic fixed point, i.e.* $f_0(z) = z + a_2z^2 + a_3z^3 + \cdots$, *there exist attracting Fatou coordinate* Φ_{attr} *and repelling Fatou coordinate* Φ_{rep}*. These Fatou coordinates are unique up to the addition of constants.*

When f is global (rational map or entire), the attracting Fatou coordinate has an extension $\Phi_{attr} : \mathcal{B}(\text{parabolic basin}) \to \widehat{\mathbb{C}}$ with equation $\Phi_{attr}(f_0(z)) = \Phi_{attr}(z) + 1$ and the inverse of repelling Fatou coordinate has an extension $\Psi_{rep} = \Phi_{rep}^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$ with equation $\Psi_{rep}(z + 1) = f_0 \circ \Psi_{rep}(z) + 1$.

Definition. The horn map is defined to be $E_{f_0}(z) = \Phi_{attr} \circ \Phi_{rep}^{-1} : \Phi_{rep}(\mathcal{B}) \to \mathbb{C}$. The domain contains some upper and lower half planes $\{\operatorname{Im} z > L\}$ and $\{\operatorname{Im} z < -L\}$. It commutes with T, i.e., $E_f \circ T = T \circ E_f$, which means that $E_f(z) - z$ is periodic with period 1. So it has Fourier expansions with only non-negative indices in the upper half and with only non-positive indices in the lower half. The constant term differs by $2\pi iA$ for the upper and lower expansions.

L_σ and $T_\sigma \circ E_f$ are semi-conjugate to each other.

2 Perturbation of parabolic fixed points and Fatou coordinates

We now perturb $f_0(z) = z + z^2 + a_3z^3 + \cdot$ to a nearby f, so that the fixed point $z = 0$ bifurcate into two distinct fixed points. One can change the coordinate so that one of the fixed points is $z = 0$. (For a parameterized family, one may need to take a branched double cover of the parameter space.) f can be written as

$$f(z) = e^{2\pi i\alpha}z + O(z^2),$$

where $\alpha = \alpha(f) \in \mathbb{C}$ is close to 0. Interesting phenomena such as the discontinuity of the Julia sets are know to occur along the direction where α is close to the real axis. So we will restrict to the case $|\arg \alpha| < \frac{\pi}{2}$.

Theorem 2.1 (Existence of attracting/repelling Fatou coordinates for perturbed maps)**.** *For* f *which a small perturbation of* f_0 *with* $|\arg \alpha(f)| < \frac{\pi}{2}$*, attracting Fatou coordinate* φ_{attr} *and repelling Fatou coordinate* φ_{rep} *in certain domains can be defined under certain normalizations (which kill the ambiguity coming from the additive constants). The horn map* $E_{f_0}(z) = \Phi_{attr} \circ \Phi_{rep}^{-1}$ *is also defined in a domain containing* $\{|\operatorname{Im} z| >> 0\}$*. They are continuous with respect to* f*. For example, when* $f_0 \to f$ *with the condition* $|\arg \alpha(f)| < \frac{\pi}{2}$, $E_f \to E_{f_0}$*.*

2.1 Return maps

For perturbed maps, there are orbits going from attracting side to repelling sides. This induces an identification between the attracting and repelling Fatou coordinates up to additive constants (which is in fact a quantity going to ∞ as the perturbation gets smaller. The orbits going through the "gap" between the fixed points defines a "return map" $\tilde{\mathcal{R}}f$ on $\{|\operatorname{Im} z| \gg 0\}$. If $\tilde{\mathcal{R}}f(z) = w$, this means $\Phi_{rep}^{-1}(z)$ and $\Phi_{rep}^{-1}(w)$ are related by some iterates of f.

Theorem 2.2. *The return map of f on the (attr) cylinder $\mathbb{C}/\mathbb{Z}$ is*

$$\tilde{\mathcal{R}}f = \chi_f \circ E_f \quad \text{where } E_f \text{ is the horn map and } \chi_f(z) = z - \frac{1}{\alpha}.$$

Corollary 2.3 (Limit of iterates)**.** *If $f_n \to f_0$ with $f_n(z) = e^{2\pi i \alpha_n} z + \cdots$ such that $\frac{1}{\alpha_n} - k_n \to \beta$, $k_n \in \mathbb{N}$, then*

$$\tilde{\mathcal{R}}f_n \to E_{f_0} - \beta.$$

3 Applications of Parabolic implosion

3.1 Discontinuity of Julia sets

The existence and behavior of return maps can be used to show:

Theorem 3.1 (Discontinuity of Julia sets)**.** *When $f \to f_0$ with condition on the argument $\arg \alpha$, then $J(f)$ is discontinuous. For example, $f_n \to f_0$, then $\liminf_{n\to\infty} J(f_n) \supsetneq J(f_0)$.*

3.2 Hausdorff dimension of Julia sets and the Mandelbrot set

Using the (limit of) return maps, we can find a subset of Julia set $J_(f_n)$ for $f_n \to f_0$, which have large Hausdorff dimension. Such a subset is constructed from an IFS whose maps are iterates of f_n restricted to some subsets.

We obtained can be summarized as:

Theorem 3.2. *If a quadratic polynomial $f_c(z) = z^2 + c$ has non-degenerate 1-parabolic periodic point, then one can perturb in certain direction so that H-dim$(J_{f_n}) \to 2$. Along this sequence, there exists a hyperbolic subset $X_n \subset J(f_n)$, with H-dim$(X_n) \to 2$ where hyperbolic subset means X_n is compact, $f_n(X_n) \subset X_n$ and $|(f_n^N)'| \geq 2$ on X_n for some N.*

Go into the parameter space

Definition (Mandelbrot set)**.**

$$M = \{c \in \mathbb{C} \mid J(z^2 + c) \text{ is connected}\} = \{c \in \mathbb{C} \mid \{f_c^n(0)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

Definition. For a rational map f, define its *hyperbolic dimension* as

$$\text{hyp-dim}(f) = \sup\{\text{H-dim}(X) \mid X \text{ is a hyperbolic subset of } J(f)\}$$

By definition, hyp-dim(f) is lower semi continuous with respect to f.

What we have shown is

Theorem 3.3. *$\{c \in \partial M \mid$ hyp-dim$(f_c) > 2 - \frac{1}{n}\}$ is open and dense in ∂M.*

By Baire category theorem, we obtain for $\cap_{n=1}^{\infty}\{c \in \partial M \mid \text{hyp-dim}(f_c) > 2 - \frac{1}{n}$,

Corollary 3.4. *For a generic (hence dense) set of $c \in \partial M$, we have*

$$\text{H-dim}(J(f_c)) = \text{hyp-dim}(f_c) = 2.$$

On the other hand, using the idea of similarity between M and J (cf. Tan Lei), for example using holomorphic motion, one can show the following:

Theorem 3.5. *H-dim*$(\partial M) \geq$ *H-dim*$(J(f_c))$ *for* $c \in \partial M$.

Theorem 3.6. *H-dim*$(\partial M) = 2$.

Exponential-like map within the polynomial

In our argument, we can observe that the biggest effect is coming from the exponential function $\mathbb{C}/\mathbb{Z} \ni z \mapsto e^{2\pi iz} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Going from one cylinder to the next cylinder involves an exponential-like mapping. So we can say that there is an exponential-like map within the polynomial dynamics. Repeating this process is related to the renormalization.
Note that the Julia set of λe^z has dimension 2 (McMullen).
Q. Can one have *sin*-like map within polynomials?
(Probably No).

3.3 Renormalization

If we can repeat the process going to a new coordinates via the exponential map $z \mapsto e^{2\pi iz}$, we will be able to see the iteration of exponential-like maps. This is the idea of "Renormalization".

In fact, we have

Theorem 3.7 (Inou & S.))**.** *For some V and N, the near-parabolic renormalization $\mathcal{R}$ from*

$$\{e^{2\pi i\alpha} f : \alpha \in Irrat_N, f \in \mathcal{F}_1\} = Irrat_N \times \mathcal{F}_1$$

is well defined and expanding along α direction and uniformly contracting along $\mathcal{F}_1$ direction. Moreover $\mathcal{R}(e^{2\pi i\alpha} z + z^2)$ belong to the above set for $\alpha \in Irrat_N$.

There are applications of this results:

Buff-Chéritat (2012) Positive area Julia set for $z^2 + \exists c$.
it uses also other type of renormalization due to McMullen and Yoccoz.

Cheraghi (2018, 2019) Ergodic property of the hedgehog

Cheraghi, S. Topological model for the hedgehog

S. and Yang Fei (2024), The boundary of high type Siegel disks are Jordan curves.

There are new approach to a similar kind of renormalization by Dudko and Lyubich.