Julia sets appear quasiconformally in the Mandelbrot set

Masashi Kisaka (木坂 正史)

Graduate School of Human and Environmental Studies, Kyoto University
(京都大学 大学院人間・環境学研究科)

第 68 回函数論シンポジウム

茨城大学水戸キャンパス　茨城大学図書館ライブラリーホール

2025 年 10 月 17 日 (金), 15:35〜16:25

This article is based on the joint research [**KK1**] with Tomoki Kawahira (川平友規).

ABSTRACT. In the boundary of the Mandelbrot set, we can find quasiconformal copies of a Cantor Julia set which is a small perturbation of the Julia set of any given parameter in the boundary of the Mandelbrot set. We can also find copies that are images of the Julia sets by quasiconformal maps with dilatation arbitrarily close to 1. This answers a question by Adrian Douady. Indeed, we can specify the locations of such copies near the boundary of any small Mandelbrot set. If we zoom in the middle part of such a copy, then we can find a certain nested structure ("decoration") and finally another "smaller Mandelbrot set" appears. A similar nested structure exists in the Julia set for any parameter in the "smaller Mandelbrot set". All the parameters belonging to these quasiconformal copies in the Mandelbrot set are semihyperbolic and this leads to the fact that the set of semihyperbolic but non-Misiurewicz and non-hyperbolic parameters is dense with Hausdorff dimension 2 in the boundary of the Mandelbrot set.

1. Introduction

Let $P_c(z) := z^2 + c$ ($c \in \mathbb{C}$) and recall that its *filled Julia set* $K(P_c)$ is defined by

$$K(P_c) := \{z \in \mathbb{C} \mid \{P_c^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}$$

and its *Julia set* $J(P_c)$ is the boundary of $K(P_c)$, that is, $J(P_c) := \partial K(P_c)$. It is known that $J(P_c)$ is connected if and only if the critical orbit $\{P_c^n(0)\}_{n=0}^{\infty}$ is bounded and if $J(P_c)$ is disconnected, then it is a Cantor set. The connectedness locus of the quadratic family $\{P_c\}_{c\in\mathbb{C}}$ is the famous *Mandelbrot set* and we denote it by M:

$$M := \{c \in \mathbb{C} \mid J(P_c) \text{ is connected}\} = \{c \in \mathbb{C} \mid \{P_c^n(0)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

A parameter c is called a *Misiurewicz parameter* if the critical point 0 is strictly preperiodic, that is,

$$P_c^k(P_c^l(0)) = P_c^l(0) \quad \text{and} \quad P_c^k(P_c^{l-1}(0)) \neq P_c^{l-1}(0)$$

for some $k,\ l \in \mathbb{N} = \{1, 2, 3, \ldots\}$. For the basic knowledge of complex dynamics, we refer to [**Bea**] and [**Mi2**].

Douady et al. ([**D-BDS**]) proved the following: in a neighborhood of the cusp point $c_0 \neq 1/4$ in M, which is the root of primitive small Mandelbrot set, there is a sequence

$\{M_n\}_{n\in\mathbb{N}}$ of small quasiconformal copies of M tending to c_0. Moreover each M_n is encaged in a nested sequence of sets which are quasiconformally homeomorphic to the preimage of $J(P_{1/4+\eta})$ (for $\eta > 0$ small) by $z \mapsto z^{2^m}$ for $m \geq 0$ and accumulate on M_n.

In this paper, firstly we generalize part of their results (Main Theorem, Theorems A and C). Actually this kind of phenomena can be observed not only in a small neighborhood of the cusp of a primitive small Mandelbrot set, that is, the point corresponding to a parabolic parameter $1/4 \in \partial M$, but also in every neighborhood of a point corresponding to any $c_0 \in \partial M$ in a small Mandelbrot set. (For example, $c_0 = 1/4 \in \partial M$ can be replaced by a Misiurewicz parameter $c_0 = i \in \partial M$ or a parabolic parameter $c_0 = -3/4 \in \partial M$ etc.) More precisely, we show the following: take any small Mandelbrot set M_{s_0} (Figure 1-(1))) and zoom in the neighborhood of $c_1 = s_0 \perp c_0 \in \partial M_{s_0}$ corresponding to $c_0 \in \partial M$ (Figure 1-(2) to (6))). Then we can find a subset $J' \subset \partial M$ which looks very similar to $J(P_{c_0})$ (Figure 1–(6)). Zoom in further, then this J' turns out to be similar to $J(P_{c_0+\eta})$ rather than $J(P_{c_0})$, where $|\eta|$ is very small and $c_0+\eta \notin M$, because J' looks disconnected (Figure 1–(8), (9)). Furthermore, as we further zoom in the middle part of J', we can see a nested structure which is very similar to the iterated preimages of $J(P_{c_0+\eta})$ by $z \mapsto z^2$ (we call these a *decoration*) (Figure 1–(10), (12), (14)) and finally another smaller Mandelbrot set M_{s_1} appears (Figure 1–(15)). This is achieved by showing that M_{s_0} and its decoration are the images of a certain model set by quasiconformal maps (Theorem A). Also we show that for some of these quasiconformal maps the dilatations can be made arbitrarily close to 1 (Main Theorem and Theorem C). This answers the first part of the "Final remarks" in [**D-BDS**, p.35].

Secondly we show the following result for filled Julia sets (Theorems B and D): take a parameter $s_1 \perp c$ ($c \in M$) from the above smaller Mandelbrot set M_{s_1} and look at the filled Julia set $K(P_{s_1 \perp c})$ and its zooms around the neighborhood of $0 \in K(P_{s_1 \perp c})$. Then we can observe a very similar nested structure to what we saw as zooming in the middle part of the set $J' \subset \partial M$ (see Figure 2). This is again explained by showing that these structures are the images of a certain model set by quasiconformal maps (Theorem B). Also we show that for some of the quasiconformal maps dilatations can be made arbitrarily close to 1 (Theorem D).

Finally we show that all the parameters belonging to the decorations are semihyperbolic and also the set of semihyperbolic but non-Misiurewicz and non-hyperbolic parameters are dense in the boundary of the Mandelbrot set (Corollary E). This together with Theorem C leads to the following direct and intuitive explanation for the fact that the Hausdorff dimension of ∂M is equal to 2, which is a famous result by Shishikura ([**Sh**]): this is because we can find a lot of almost conformal images of Cantor quadratic Julia sets whose Hausdorff dimension are arbitrarily close to 2, which are known to exist ([**Sh**]).

2. The Model Sets and the Statements of the Results

Notation. We use the following notation for disks and annuli:

$$D(R) := \{z \in \mathbb{C} \mid |z| < R\}, \quad \mathbb{D} := D(1), \quad D(\alpha, R) := \{z \in \mathbb{C} \mid |z - \alpha| < R\},$$
$$A(r, R) := \{z \in \mathbb{C} \mid r < |z| < R\} \quad (0 < r < R).$$

We mostly follow Douady's notations in [**D-BDS**] in the following.

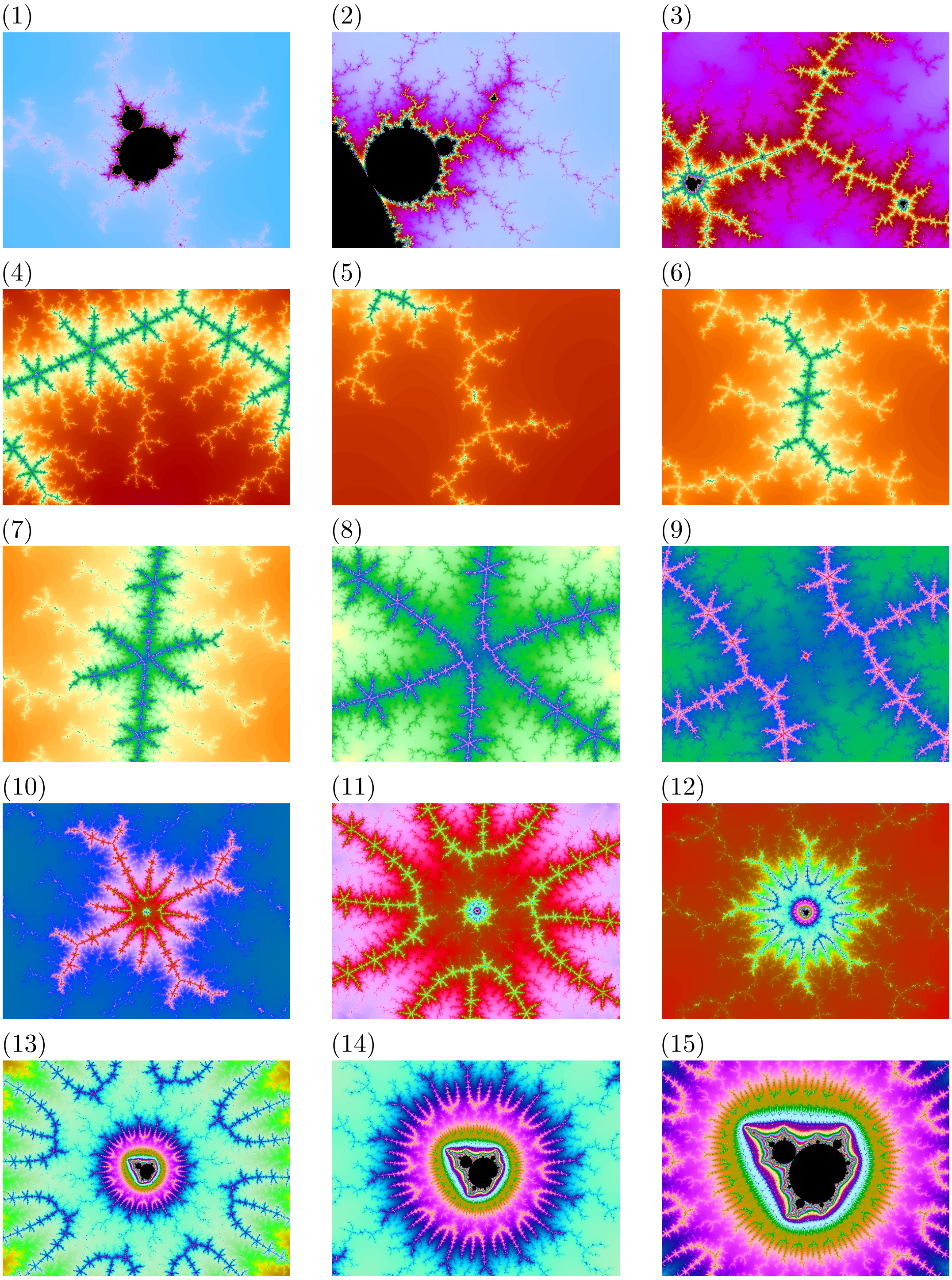

FIGURE 1. Zooms around a Misiurewicz point $c_1 = s_0 \perp c_0$ in a primitive small Mandelbrot set M_{s_0}, where c_0 is a Misiurewicz parameter satisfying $P_{c_0}(P_{c_0}^4(0)) = P_{c_0}^4(0)$. After a sequence of nested structures, another smaller Mandelbrot set M_{s_1} appears in (15). Here, $s_0 \approx 0.3591071125276155 + 0.6423830938166145i$, $c_0 \approx -0.1010963638456221 + 0.9562865108091415i$, $c_1 \approx 0.3626697754647427 + 0.6450273437137847i$ and $s_1 \approx 0.3626684938191616 + 0.6450238859863952i$. The widths of the figures (1) and (15) are about $10^{-1.5}$ and $10^{-11.9}$, respectively.

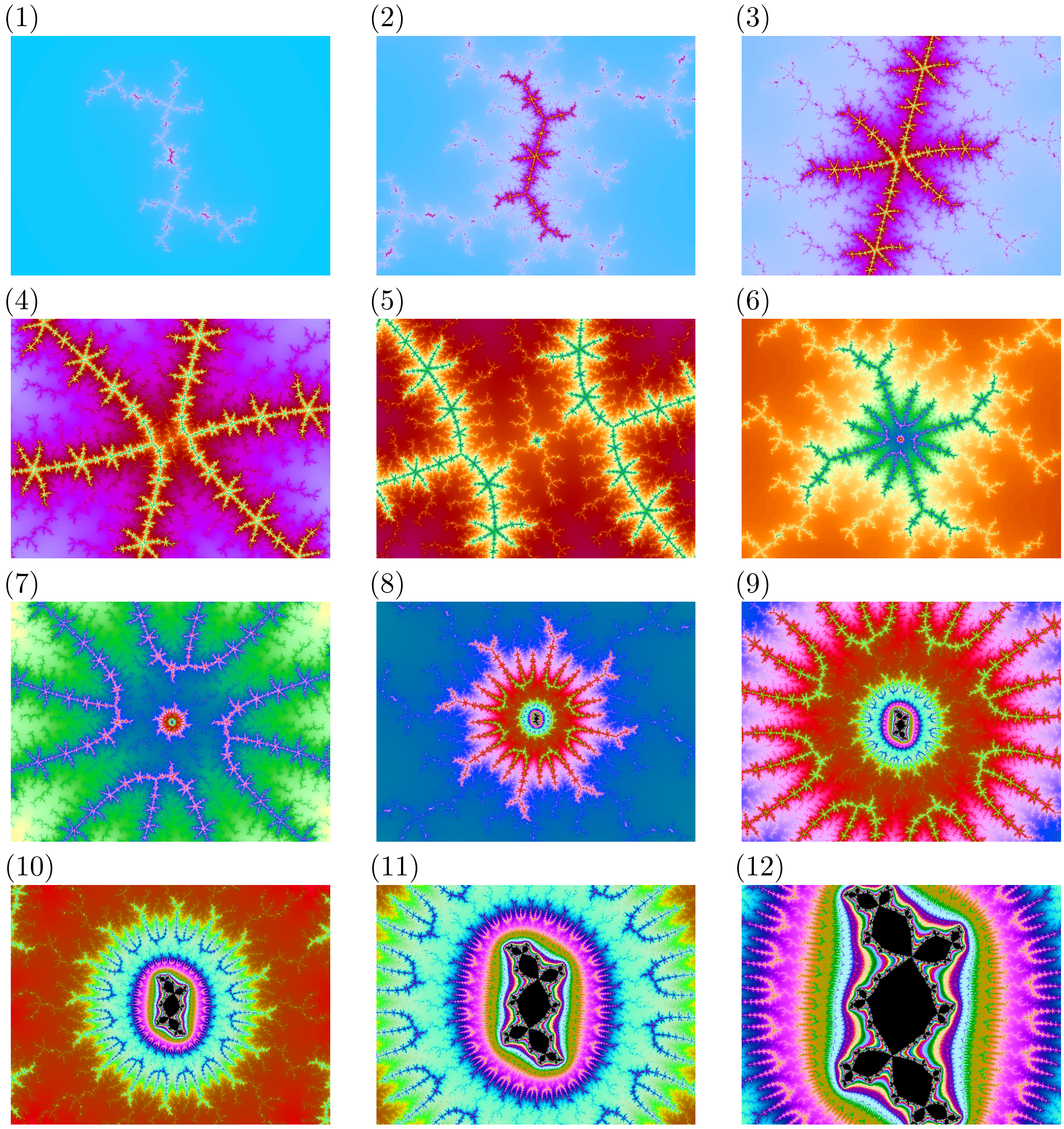

FIGURE 2. Zooms around the critical point 0 in $K(P_{s_1 \perp c})$ for $s_1 \perp c$ in M_{s_1}, which is the smaller Mandelbrot set in Figure 1–(15) and $c \in M$ is the parameter for the Douady rabbit. $s_1 \approx 0.3626684938191616 + 0.6450238859863952i$, $c \approx -0.12256 + 0.74486i$ and $s_1 \perp c \approx 0.3626684938192285 + 0.6450238859865394i$.

Models. Let $\sigma \notin M$. In this case $J(P_\sigma)$ is a Cantor set which does not contain 0. Now take two positive numbers ρ' and ρ such that

$$J(P_\sigma) \subset A(\rho', \rho) \quad (\rho' < \rho).$$

We define the *rescaled Julia set* $\Gamma_0(\sigma)$ by

$$\Gamma_0(\sigma) := J(P_\sigma) \times \frac{\rho}{(\rho')^2} = \left\{ \frac{\rho}{(\rho')^2} z \;\middle|\; z \in J(P_\sigma) \right\}$$

such that $\Gamma_0(\sigma)$ is contained in the annulus $A(R, R^2)$ with $R := \rho/\rho'$. In [**D-BDS**], Douady used the radii of the form $\rho' = R^{-1/2}$ and $\rho = R^{1/2}$ for some $R > 1$ such that $\Gamma(\sigma) = J(P_\sigma) \times R^{3/2}$ is contained in $A(R, R^2)$. In this paper, however, we need more flexibility when we are concerned with the dilatation.

Let $\Gamma_m(\sigma)$ ($m \in \mathbb{N}$) be the inverse image of $\Gamma_0(\sigma)$ by $z \mapsto z^{2^m}$, the m-th iteration of $z \mapsto z^2$. Then $\Gamma_m(\sigma)$ ($m = 0,\ 1,\ 2,\ \ldots$) are mutually disjoint, because we have

$$\Gamma_0(\sigma) \subset A(R, R^2),\ \Gamma_1(\sigma) \subset A(R^{1/2}, R),\ \Gamma_2(\sigma) \subset A(R^{1/4}, R^{1/2}), \cdots .$$

For another parameter $c \in M$, let $\Phi_c : \mathbb{C} \smallsetminus K(P_c) \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ be the Böttcher coordinate, i.e., Φ_c is a conformal isomorphism with $\Phi_c(P_c(z)) = (\Phi_c(z))^2$. (In this paper, following [**D-BDS**], we use the term *conformal isomorphism*, or simply *isomorphism*, to refer to a biholomorphic mapping.) Let $\Phi_M : \mathbb{C} \smallsetminus M \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ be the conformal isomorphism defined by $\Phi_M(c) := \Phi_c(c)$, which satisfies the condition $\Phi_M(c)/c \to 1$ as $|c| \to \infty$ (see [**DH1**]). Now define the *model sets* $\mathcal{M}(\sigma)$ and $\mathcal{K}_c(\sigma)$ as follows (see Figure 3):

$$\mathcal{M}(\sigma) := M \cup \Phi_M^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(\sigma)\Big), \quad \mathcal{K}_c(\sigma) := K(P_c) \cup \Phi_c^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(\sigma)\Big).$$

We call $\mathcal{M}(\sigma)$ a *decorated Mandelbrot set*, $\mathcal{M}(\sigma) \smallsetminus M = \Phi_M^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(\sigma)\Big)$ its *decoration* and $M \subset \mathcal{M}(\sigma)$ the *main Mandelbrot set* of $\mathcal{M}(\sigma)$. Also we call $\mathcal{K}_c(\sigma)$ a *decorated filled Julia set* and $\mathcal{K}_c(\sigma) \smallsetminus K(P_c) = \Phi_c^{-1}\Big(\bigcup_{m=0}^{\infty} \Gamma_m(\sigma)\Big)$ its *decoration.* We will apply the same terminologies to the images of $\mathcal{M}(\sigma)$ or $\mathcal{K}_c(\sigma)$ by quasiconformal maps.

Since the sets $\Gamma_m(\sigma)$, $\mathcal{M}(\sigma)$ and $\mathcal{K}_c(\sigma)$ depend on the choice of ρ' and ρ, we denote them by $\Gamma_m(\sigma)_{\rho',\rho}$, $\mathcal{M}(\sigma)_{\rho',\rho}$, and $\mathcal{K}_c(\sigma)_{\rho',\rho}$ respectively when we emphasize the dependence.

Quasiconformal copies. Let X and Y be non-empty compact sets in $\mathbb{C}$. We say that X *appears (K-)quasiconformally in* Y or Y *contains a (K-)quasiconformal copy of* X if there exists a (K-)quasiconformal map χ on a neighborhood of X such that $\chi(X) \subset Y$ and $\chi(\partial X) \subset \partial Y$. Note that the condition $\chi(\partial X) \subset \partial Y$ is to exclude the case $\chi(X) \subset \mathrm{int}(Y)$.

Main Theorem. *For any $c_0 \in \partial M$ and any small $\varepsilon > 0$ and $\kappa > 0$, there exists a parameter*

$$\sigma \in D(c_0, \varepsilon) \smallsetminus M$$

such that M contains a $(1+\kappa)$-quasiconformal copy of $\mathcal{M}(\sigma) = \mathcal{M}(\sigma)_{\rho',\rho}$ for some ρ' and ρ. Moreover, one can find such a copy in any open disk intersecting with ∂M.

Since $\mathcal{M}(\sigma)$ contains the rescaled Julia set $\Gamma_0(\sigma) = J(P_\sigma) \times \rho/(\rho')^2$, we may say that *the Julia set $J(P_\sigma)$ appears almost conformally in M.*

Note that if $K(P_{c_0})$ has empty interior (i.e., P_{c_0} has no parabolic basins nor Siegel disks), then $J(P_\sigma)$ tends to $J(P_{c_0})$ in the Hausdorff topology as $\sigma \to c_0$. Even in the case when the interior of $K(P_{c_0})$ is non-empty, $J(P_\sigma)$ is contained in the η-neighborhood of $K(P_{c_0})$, and the η-neighborhood of $J(P_\sigma)$ contains $J(P_{c_0})$ for any given $\eta > 0$ if σ is sufficiently close to c_0. See [**Do**]. This explains why we can find structures that resemble the Julia set $J(P_{c_0})$ everywhere in the boundary of the Mandelbrot set.

The Main Theorem will be restated and proved as Theorem C below, with a slightly modified formulation.

Small Mandelbrot sets. For a given $c_0 \in \partial M$, the locations of quasiconformal copies of $\mathcal{M}(\sigma)$ in M as in the Main Theorem can be more precisely identified by introducing the concept of the *small Mandelbrot set.* Let $s_0 \neq 0$ be a superattracting parameter such that the period of the critical point 0 is more than one. By the Douady-Hubbard tuning

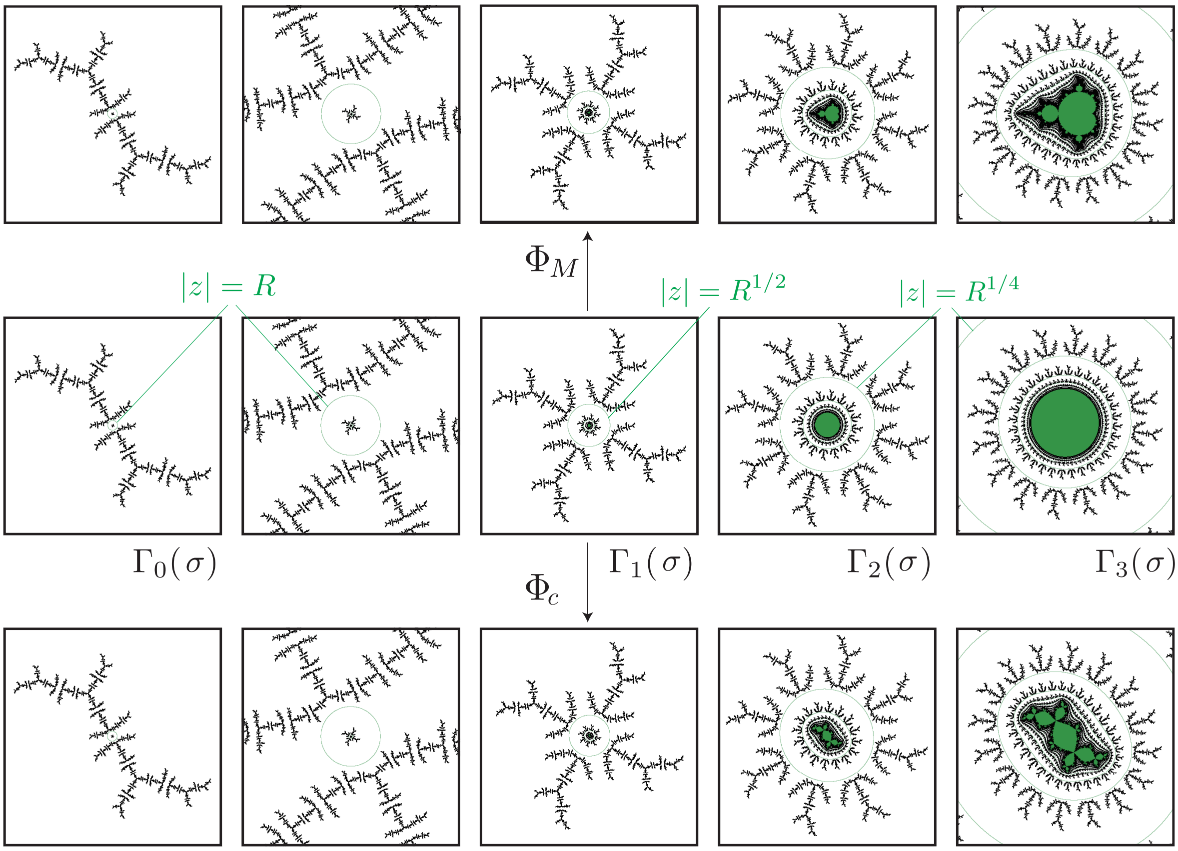

FIGURE 3. The first row depicts the decorated Mandelbrot set $\mathcal{M}(\sigma)$ for $\sigma = -0.10 + 0.97\,i$ (close to the Misiurewicz parameter $c_0 \approx -0.1011 + 0.9563\,i$, the landing point of the external ray of angle $11/56$) and $R = 220$. The second row depicts the set $\bigcup_{m \geq 0} \Gamma_m(\sigma)$. The third row depicts the decorated filled Julia set $\mathcal{K}_c(\sigma)$ for $c \approx -0.123 + 0.745\,i$ (close to the rabbit).

theorem (see [**H**, Théorème 1 du Modulation] and [**Mi1**]), there exists a unique compact subset M_{s_0} of M associated with a canonical homeomorphism $\chi_{s_0} : M_{s_0} \to M$ such that $\chi_{s_0}(s_0) = 0$. See Figure 4. We also denote M_{s_0} by $s_0 \perp M$ and call it the *small Mandelbrot set with center* s_0. Similarly, for $c_0 \in M$, let $s_0 \perp c_0$ denote the parameter $\chi_{s_0}^{-1}(c_0)$ in M_{s_0}.

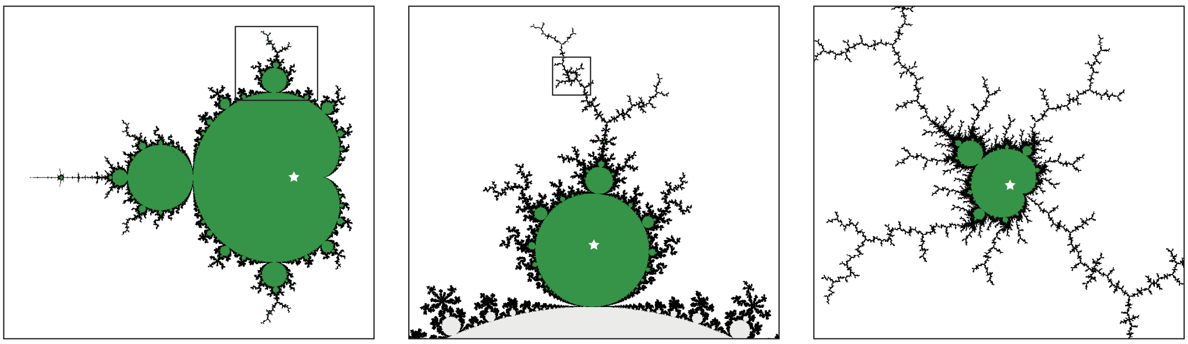

FIGURE 4. The original Mandelbrot set (left), a *satellite* small Mandelbrot set (middle), and a *primitive* small Mandelbrot set (right). See Section 4 for the dichotomy between satellite and primitive small Mandelbrot sets. The stars indicate the central superattracting parameters.

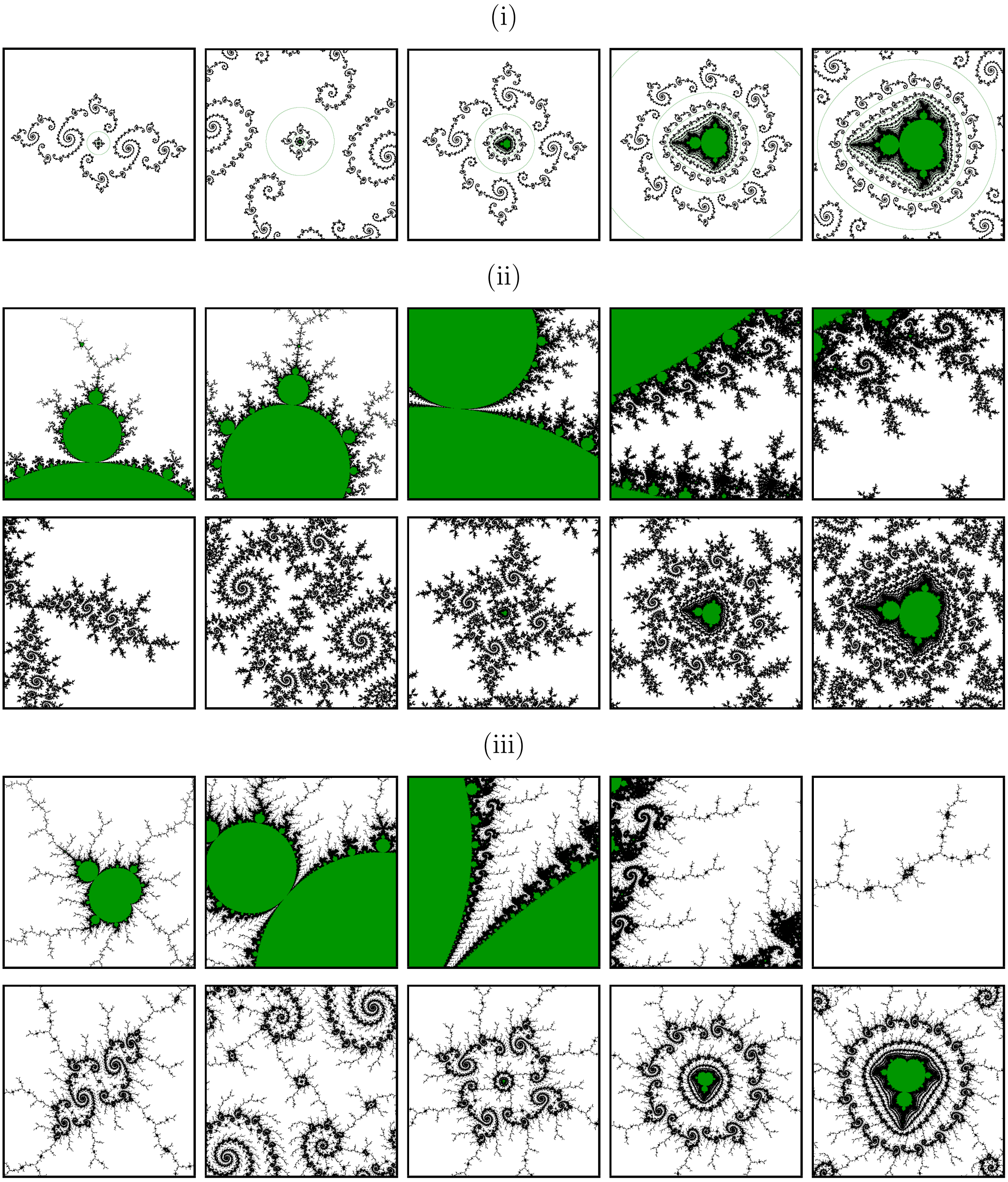

Figure 5. (i): The decorated Mandelbrot set $\mathcal{M}(\sigma)$ for $\sigma = -0.77+0.18i$ (close to the parabolic parameter $c_0 = -0.75$. (ii) and (iii): Embedded quasiconformal copies of $\mathcal{M}(\sigma)$ above near the satellite/primitive small Mandelbrot sets in Figure 4.

First we show the following theorem *without* including dilatation estimates, as an extension of the result by Douady et al. [**D-BDS**]:

Theorem A (Julia sets appear quasiconformally in M). *Let c_0 be any parameter in ∂M, and M_{s_0} be any small Mandelbrot set with center $s_0 \neq 0$. Let $c_1 := s_0 \perp c_0 \in \partial M_{s_0}$. Then for any $\varepsilon > 0$ and $\varepsilon' > 0$, there exists an $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$ and $c_0 + \eta \notin M$ such* *that $\mathcal{M}(c_0+\eta)$ appears quasiconformally in $M \cap \overline{D(c_1,\varepsilon')}$. In particular, the Cantor Julia set $J(P_{c_0+\eta})$ appears quasiconformally in M.*

This theorem provides an explanation for the images shown in Figure 1. The proof generalizes the framework developed by Douady et al. [**D-BDS**], which only addresses the case where $c_0 = 1/4$ and focuses on the small Mandelbrot set M_{s_0} of particular type. We also substitute their parabolic implosion technique with a shooting technique at Misiurewicz parameters in our approach.

Next theorem is a dynamical counterpart of Theorem A:

Theorem B (Julia sets appear quasiconformally in K). *Let $\mathcal{M}' \subset M$ be the quasiconformal copy of $\mathcal{M}(c_0+\eta)$ in Theorem A, and M_{s_1} be the main Mandelbrot set of $\mathcal{M}'$. Then we have:*

(1) *For every $c \in M$, the set $\mathcal{K}_c(c_0+\eta)$ appears quasiconformally in $K(P_{s_1 \perp c})$, where $s_1 \perp c \in M_{s_1}$. In particular, the Cantor Julia set $J(P_{c_0+\eta})$ appears quasiconformally in $K(P_{s_1 \perp c})$.*
(2) *There exists a neighborhood W of $\mathcal{M}'$ such that the Cantor Julia set $J(P_{c_0+\eta})$ appears quasiconformally in $K(P_\sigma)$ for any $\sigma \in W$.*

Remark that the decoration of $\mathcal{K}_c(c_0+\eta)$ is conformally the same as that of $\mathcal{M}(c_0+\eta)$. Item (1) of this theorem provides an explanation for the images shown in Figure 2. Item (2) also explains the nested and complicated structure of ∂M presented later in Figure 6. See Remark (2) below for more details.

Almost conformal copies. The following is a more detailed version of the Main Theorem:

Theorem C (Almost conformal copies in M). *Let c_0 be any parameter in ∂M and B any open disk intersecting with ∂M. Then for any small $\varepsilon > 0$ and $\kappa > 0$, there exist an $\eta \in \mathbb{C}$ with $|\eta| < \varepsilon$ and two positive numbers ρ' and ρ with $\rho' < \rho$ such that $c_0+\eta \notin M$ and $\mathcal{M}(c_0+\eta)_{\rho',\rho}$ appears $(1+\kappa)$-quasiconformally in $M \cap \overline{B}$. In particular, $M \cap \overline{B}$ contains a $(1+\kappa)$-quasiconformal copy of the Cantor Julia set $J(P_{c_0+\eta})$.*

The proof relies on the method developed for Theorem A and on careful control of the dilatations. As a by-product of the proof, we obtain a version of Theorem B corresponding to Theorem C:

Theorem D (Almost conformal copies in K). *Under the assumption of Theorem C, the same statement as Theorem B holds with $(1+\kappa)$-quasiconformality.*

Semihyperbolicity and Hausdorff dimension. A quadratic polynomial $P_c(z) = z^2+c$ (or the parameter c) is called *semihyperbolic* if

(1) the critical point 0 is non-recurrent, that is, $0 \notin \omega(0)$, where $\omega(0)$ is the ω-limit set of the critical point 0 and
(2) P_c has no parabolic periodic points.

If P_c is semihyperbolic, then it is known that it has no Siegel disks nor Cremer points ([**Ma**], [**CJY**]). Moreover, the Julia set $J(P_c)$ is measure 0 from the results by Lyubich

([**L1**]) and Shishikura (unpublished), also by [**CJY**, p.2, Theorem 1.1]. Thus the semihyperbolic dynamics is relatively understandable. It is easy to see that if P_c is hyperbolic then it is semihyperbolic. A well-known example of semihyperbolic but non-hyperbolic parameter c is a Misiurewicz parameter. However, it is not straightforward to construct explicit examples of semihyperbolic parameter c which is neither hyperbolic nor Misiurewicz. Next corollary shows that we can visually identify these parameters everywhere in ∂M.

Corollary E (Abundance of semihyperbolicity). *For every parameter c belonging to the quasiconformal image of the decoration of $\mathcal{M}(c_0+\eta)$ in Theorems A and C, P_c is semihyperbolic.*

It also implies a well-known fact that the set of semihyperbolic parameters that are not Misiurewicz nor hyperbolic is dense in ∂M.

Corollary E together with Theorem C explains the following famous result by Shishikura:

Theorem (Shishikura, 1998). *Let*

$$SH := \{c \in \partial M \mid P_c \text{ is semihyperbolic}\},$$

then the Hausdorff dimension of SH is 2. In particular, the Hausdorff dimension of the boundary of M is 2.

Explanation. For any $\delta > 0$, there exists an open disk D intersecting with ∂M such that all Julia sets corresponding to the parameters in D have the Hausdorff dimension at least $2-\delta$ ([**Sh**, p.231, *proof of Theorem B* and p.232, *Remark* 1.1 (iii)]). Now apply Theorem C with $c_0 \in \partial M \cap D$ and sufficiently small ε and κ such that $c_0+\eta \in D \setminus M$ and thus we can find quasiconformal copies of $J(P_{c_0+\eta})$ with Hausdorff dimension at least $2-2\delta$ everywhere in ∂M. Then by Corollary E it follows that we can find a subset of ∂M with Hausdorff dimension arbitrarily close to 2 and consisting of semihyperbolic parameters. This implies that $\dim_H(SH) = 2$. ■

The novelty of our explanation is that it enables the visual identification of structures in ∂M with high Hausdorff dimensions.

Remark. (1) A similar result to (1) of Theorem B still holds even when $c \in \mathbb{C} \setminus M$ is sufficiently close to M. Indeed, if $c \in \mathbb{C} \setminus M$ satisfies $s_1 \perp c \in W \setminus M_{s_1}$, where W is the neighborhood of $\mathcal{M}'$ given in (2) of Theorem B, then the model set $\mathcal{K}_c(c_0+\eta)$ can be defined by modifying the original definition for $c \in M$. We can also prove that a $\mathcal{K}_c(c_0+\eta)$ appears quasiconformally in $K(P_{s_1\perp c})$. We omit the details.

(2) Take a small Mandelbrot set M_{s_1} (e.g. Figure 1–(15) = Figure 6–(1)) and another parameter $c_* \in \partial M$ (e.g. $c_* = 1/4$ in Figure 6) and zoom in the neighborhood of $s_1 \perp c_*$. Then we see much more complicated structure than we expected as follows: according to Theorem A, by replacing s_0 with s_1 and c_0 with c_*, it says that $\mathcal{M}(c_*+\eta)$ appears quasiconformally in $D(s_1 \perp c_*, \varepsilon')$. This means that as we zoom in, we first see a quasiconformal image of $J(P_{c_*+\eta_*})$, say $\widetilde{J}_{c_*+\eta_*}$ (e.g. "broken cauliflower", when $c_* = 1/4$). But in reality as we zoom in, what we first see is a $\widetilde{J}_{c_0+\eta_0}$ (e.g. "broken dendrite". See Figure 6–(5)). This seems to contradict with Theorem A, but actually it does not. As we zoom in further in the middle part of $\widetilde{J}_{c_0+\eta_0}$, we see iterated preimages of $\widetilde{J}_{c_0+\eta_0}$ by

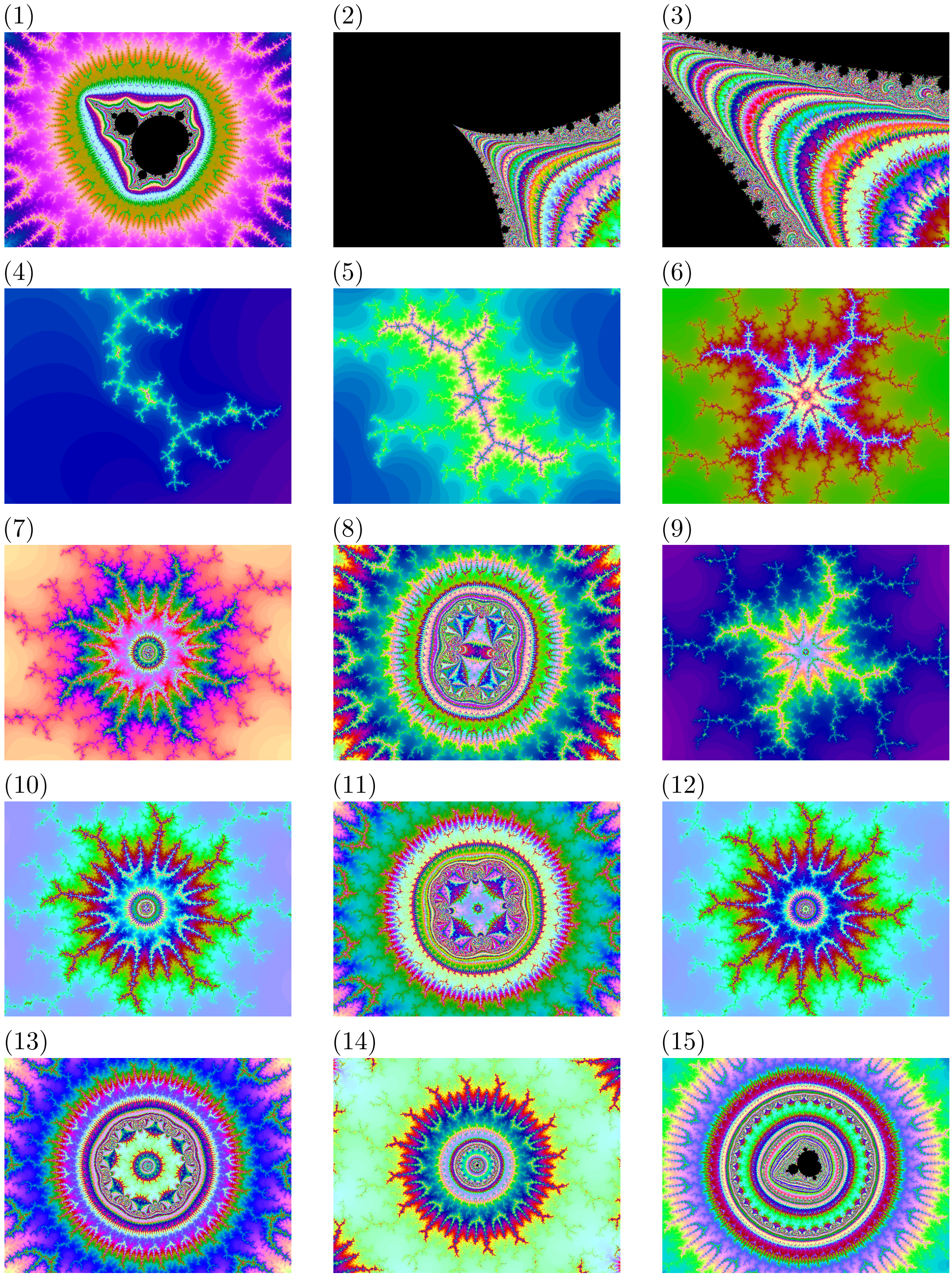

FIGURE 6. Zooms around a parabolic point $s_1 \perp c_1$ in a primitive small Mandelbrot set M_{s_1}. After a sequence of complicated nested structures, another smaller Mandelbrot set M_{s_2} appears ((15)).

$z \mapsto z^2$ (Figure 6–(6), (7)) and then $\widetilde{J}_{c_*+\eta_*}$ appears (Figure 6–(8)). After that we see again iterated preimages of $\widetilde{J}_{c_0+\eta_0}$ by $z \mapsto z^2$ (Figure 6–(9), (10)) and then a once iterated preimage of $\widetilde{J}_{c_*+\eta_*}$ appears (Figure 6–(11)). This structure continues and finally, we see a smaller Mandelbrot set, say M_{s_2} (Figure 6–(15)). We can explain this phenomena as follows: what we see in the series of magnifications above is a quasiconformal image of

$\mathcal{M}(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0))$, where

$$\begin{aligned}\mathcal{M}(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)) &:= M \cup \Phi_M^{-1}\Big(\bigcup_{m=0}^{\infty}\Gamma_m(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0))\Big),\\ \Gamma_m(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)) &:= \text{the inverse image of } \Gamma_0(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)) \text{ by } z \mapsto z^{2^m}.\end{aligned}$$

Here $\mathcal{M}(\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0))$ is obtained just by replacing $J(P_\sigma)$ with $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ in the definition of $\mathcal{M}(\sigma)$. Although $c_*+\eta_* \notin M$, $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ can be defined in the similar manner. See the Remark (1) above. So what we first see as we zoom in the neighborhood of $s_1 \perp c_*$ is a quasiconformal image of $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$, whose outer most part is $\widetilde{J}_{c_0+\eta_0}$ (= broken dendrite) and inner most part is $\widetilde{J}_{c_*+\eta_*}$ (= broken cauliflower). As we zoom in further, we see quasiconformal image of the preimage of $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ by $z \mapsto z^2$, whose inner most part is a once iterated preimage of $\widetilde{J}_{c_0+\eta_0}$. After we see successive preimages of $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ by $z \mapsto z^2$, a much smaller Mandelbrot set M_{s_2} finally appears. Since $\mathcal{K}_{c_*+\eta_*}(c_0+\eta_0)$ itself has a nested structure, the total picture has this very complicated structure. The proof is similar to that of Theorem A.

(3) In [**KK2**] we present an alternative proof of Theorem A by using the parabolic implosion technique. This proof can be easily adopted to the proof of Theorem C.

References

[**A**] L.V. Ahlfors, *Complex Analysis, third ed.*, McGraw-Hill Book Co., New York, 1978.

[**Bea**] A.F. Beardon, *Iteration of Rational Functions*, Springer Verlag, New York, Berlin and Heidelberg, 1991.

[**BR**] L. Bers and H.L. Royden, Holomorphic families of injections, *Acta Math.* **157** (1986), 259–286.

[**Ber**] F. Berteloot, La mesure harmonique de l'ensemble de Mandelbrot contient celles des ensembles de Julia, *J. Geom. Anal.*, **33** (2023), no. 183.

[**BH**] X. Buff and C. Henriksen, Julia sets in parameter spaces, *Comm. Math. Phys.* **220** (2001), no. 2, 333–375.

[**CJY**] L. Carleson, P.W. Jones and J.-C. Yoccoz, Julia and John, *Bol. Soc. Brasil. Mat. (N.S.)* **25** (1994), no. 1, 1–30.

[**CRY**] D. Cornell, C. Rojas and M. Yampolsky, Non computable Mandelbrot-like set for a one-parameter complex family, *Inform. and Comput.* **262** (2018), part 1, 110–122.

[**CT**] G. Cui and L. Tan, Hyperbolic-parabolic deformations of rational maps, *Science China Math.* **61** (2018) 2157–2220.

[**Du**] P.L. Duren, *Univalent Functions*, Springer-Verlag, 1983.

[**Do**] A. Douady, Does a Julia set depend continuously on the polynomial? *Complex Dynamical Systems (Cincinnati, OH, 1994)*, 91–138, *Proc. Sympos. Appl. Math.,*, **49** American Mathematical Society, Providence, RI, 1994.

[**D-BDS**] A. Douady, X. Buff, R. Devaney and P. Sentenac, Baby Mandelbrot sets are born in cauliflowers, in *The Mandelbrot set, Theme and Variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 19–36.

[**DH1**] A. Douady and J.H. Hubbard, Etude dynamique des polynômes complexes I & II, *Publ. Math. Orsay, Université de Paris-Sud*, Département de Mathématiques, Orsay, 1984/85, 84-2, 85-4.

[**DH2**] A. Douady and J. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.* **18** (1985), 287–343.

[**EE**] J.-P. Eckmann and H. Epstein, Scaling of Mandelbrot sets generated by critical point preperiodicity, *Comm. Math. Phys.*, **101** (1985), 283–289.

[**G**] D. Gaidashev, Cylinder renormalization for Siegel discs and a constructive measurable Riemann mapping theorem, *Nonlinearity*, **20** (2007), 713–741.

[**GS**] J. Graczyk and G. Świątek, Analytic structures and harmonic measure at bifurcation locus, *Adv. Math.*, **421** (2023), 108998.

[**H**] P. Haïssinsky, Modulation dans l'ensemble de Mandelbrot, in *The Mandelbrot set, Theme and Variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 37–66.

[**IT**] Y. Imayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*. Springer, 1992.

[**J**] W. Jung, personal communication, 2015. See `http://www.mndynamics.com/indexp.html` (his web page) and also `http://www.mrob.com/pub/muency/embeddedjuliaset.html`.

[**K**] T. Kawahira, Quatre applications du lemme de Zalcman à la dynamique complexe, *J. Anal. Math.* **124** (2014), 309–336.

[**KK1**] T. Kawahira and M. Kisaka, Julia sets appear quasiconformally in the Mandelbrot set, *Preprint*, `arXiv:1804.00176v2`, posted on 15 January 2024.

[**KK2**] T. Kawahira and M. Kisaka, Julia sets appear quasiconformally in the Mandelbrot set, II: A parabolic proof, *Submitted*, available on request.

[**L1**] M. Lyubich, On the Lebesgue measure of the Julia set of a quadratic polynomial, *Preprint*, IMS at Stony Brook, #1991/10.

[**L2**] M. Lyubich, Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture, *Ann. of Math.* (2) **149** (1999), no. 2, 319–420.

[**L3**] M. Lyubich, *Conformal Geometry and Dynamics of Quadratic Polynomials, vol I–II.* Available in his web page (`http://www.math.stonybrook.edu/~mlyubich/book.pdf`).

[**LM**] M. Lyubich and Y. Minsky, Laminations in holomorphic dynamics, *J. Differential Geom.* **47** (1997), no. 1, 17–94.

[**Ma**] R. Mañé, On a theorem of Fatou, *Bol. Soc. Brasil. Mat. (N.S.)* **24** No.1 (1993), 1–11.

[**Mc1**] C.T. McMullen, *Renormalization and Complex Dynamics,* Ann. of Math. Studies **135**, Princeton Univ. Press, 1994.

[**Mc2**] C.T. McMullen, The Mandelbrot set is universal, in *The Mandelbrot set, Theme and Variations*, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, Cambridge (2000), 1–17.

[**Mi1**] J. Milnor, Periodic orbits, externals rays and the Mandelbrot set: an expository account, *Géométrie Complexe et Systèmes Dynamiques* (Orsay, 1995), Astérisque **261** (2000), xiii, 277–333.

[**Mi2**] J. Milnor, *Dynamics in One Complex Variable, third ed.*, Annals of Mathematics Studies, vol. **160**, Princeton University Press, Princeton, NJ, 2006.

[**MNTU**] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic Dynamics,* Translated from the 1995 Japanese original and revised by the authors. Cambridge Studies in Advanced Mathematics **66**, Cambridge University Press, Cambridge, 2000.

[**MTU**] S. Morosawa, M. Taniguchi and T. Ueda, *Holomorphic Dynamics* (in Japanese), Baifuukan (publisher), 1995.

[**PS**] H.-O. Peitgen and D. Saupe (eds.), *The Science of Fractal Images*, Springer-Verlag, New York, 1988.

[**R**] J. Rivera-Letelier, On the continuity of Hausdorff dimension of Julia sets and similarity between the Mandelbrot set and Julia sets, *Fund. Math.* **170** (2001), no. 3, 287–317.

[**Sh**] M. Shishikura, The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets, *Ann. of Math.* (2) **147** (1998), no. 2, 225–267.

[**Sł**] Z. Słodkowski, Holomorphic motions and polynomial hulls, *Proc. Amer. Math. Soc.* **111** (2) (1991), 347–355.

[**T**] L. Tan, Similarity between the Mandelbrot set and Julia sets, *Comm. Math. Phys.* **134** (1990), no. 3, 587–617.